Big Data and A General Theory of Concept Lattice

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Common Problem in Big Data

- Given a Formal Context at scale, how to find the categorization, membership prediction and all logic implications?

The Formal Context $F(G, M)$

G \ M	a	b	c	d	e
1	X		X	X	X
2	X		X		
3		X			X
4		X			X
5	X				
6	X	X			X

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Looking for General Theory

- In the Big Data Analytics, it was recognized that prediction and description are two generic goals
- Prediction often appears with the use of attributes to predict the membership of a particular object in some object set with similar attributes.
- Description looks for the attributes that describes the object, often it involves identifying a set of attributes that are shared by all objects in the set.
- Is there a general theory to achieve these goals?
- Formal Concept Analysis: Classify object classes according to the common property
- Rough Set Lattice: Discern an object class via its property others don't have

Formal Context

A { binary relation / map from element to set } $\underline{\underline{R}}$ defined for

Objects $G=\{1,2,3,4,5,6\}$ and Attributes $M=\{a,b,c,d,e\}$:

$\forall g\in G \forall m\in M \;\; gRm \equiv mRg \;\; \left(m\in g^R \subseteq M \text{ or } g\in m^R \subseteq G\right),$

the object 1 possesses the attributes $\{a,c,d,e\}$

$1Ra \quad 1Rc \quad 1Rd \quad 1Re$

$1^R=\{a,c,d,e\}$ (an attribute set)

the attribute a is carried by the objects $\{1,2,5,6\}$

$aR1 \quad aR2 \quad aR5 \quad aR6$

$a^R=\{1,2,5,6\}$ (an object set)

The Formal Context $F(G,M)$

G \ M	a	b	c	d	e
1	X		X	X	X
2	X		X		
3		X			X
4		X			X
5	X				
6	X	X			X

Formal context { (G,M,I) }
Information system { (U,V,R) } $\mapsto F(G,M)$

A definite object /attribute collection
in which it is explicit given
whether every object carry
each referred attribute or not

R. Wille, Formal Concept Analysis as Mathematical Theory of Concepts and Concept Hierarchies in B. Ganter et al. (Eds.): Formal Concept Analysis, LNAI 3626 (2005) 133

Formal Concept Lattice

(X, Y) is called an FCL concept if $X^I = Y$ and $Y^I = X$

X is called the FCL extent and Y the FCL intent, e.g.,

$$\begin{cases} X = \{3,4,6\} \\ Y = \{b,e\} \end{cases} \text{ gives } \begin{cases} X^I = \{3,4,6\}^I = \{b,e\} \\ Y^I = \{b,e\}^I = \{3,4,6\} \end{cases}$$

* Find **ALL** the particular Object-set Attribute-set pairs (***X,Y***) in which ***X*** (the extent) is the largest object set possessing ***Y*** in common. These pairs can be ordered as a Lattice since

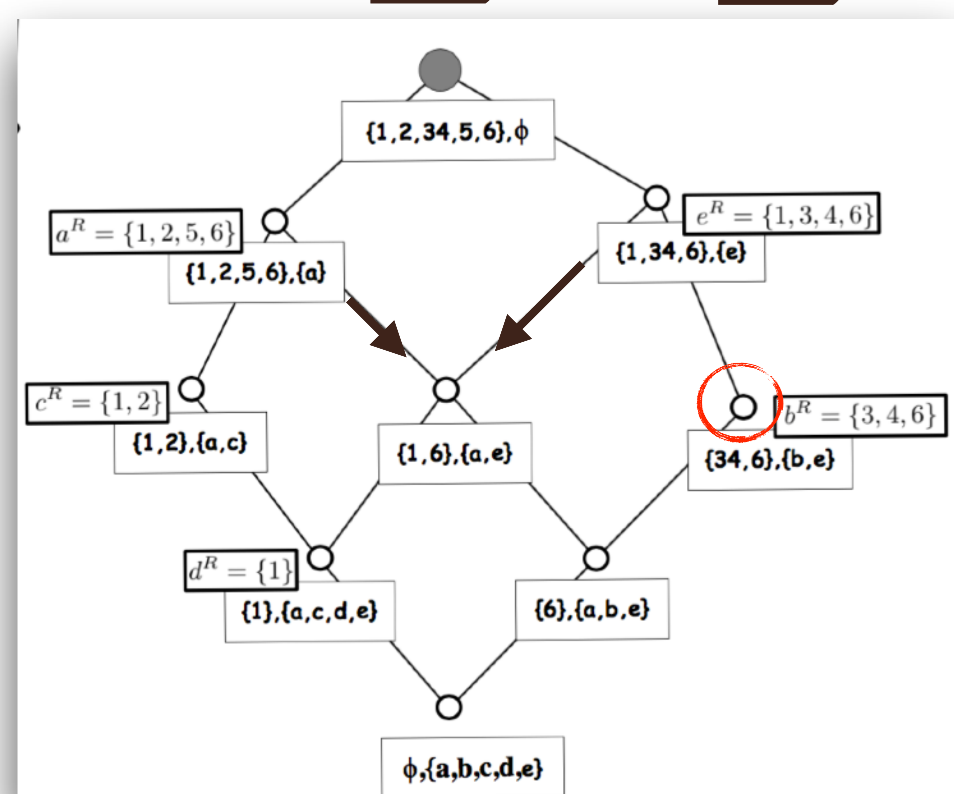
$$X_1 \subseteq X_2 \iff (X_2)^I \subseteq (X_1)^I$$

* $\forall Y \subseteq M \; (Y^I, Y^{II})$ is a formal concept for FCL

* (X_1, Y_1) and (X_2, Y_2) are nodes $\Rightarrow (X_1 \cap X_2, Y_1 \cup Y_2)$ is a node,

$$\text{e.g.,} \begin{cases} (\{1,3,4,6\},\{e\}) \\ (\{1,2,5,6\},\{a\}) \end{cases} \Rightarrow (\{1,6\},\{a,e\})$$

	a	b	c	d	e
1	X		X	X	X
2	X		X		
3		X			X
4		X			X
5	X				
6	X	X			X



Derivative Operators

Map
object set to attribute set
attribute set to object set

$$X \subseteq G \;\mapsto\; X^I = \{m \in M \mid gRm,\ \forall g \in X\} \subseteq M,$$

$$Y \subseteq M \;\mapsto\; Y^I = \{g \in G \mid gRm,\ \forall m \in Y\} \subseteq G,$$

$$X \subseteq G \;\mapsto\; X^\Box = \{m \in M \mid \forall g \in G, gRm \implies g \in X\} \subseteq M,$$

$$Y \subseteq M \;\mapsto\; Y^\Box = \{g \in G \mid \forall m \in M, gRm \implies m \in M\} \subseteq G,$$

$$X \subseteq G \;\mapsto\; X^\Diamond = \{m \in M \mid \exists g \in G, (gRm,\ g \in X)\} \subseteq M,$$

$$Y \subseteq M \;\mapsto\; Y^\Diamond = \{g \in G \mid \exists m \in M, (gRm,\ m \in M)\} \subseteq G,$$

X^I : all the attributes in which ***X*** possess in common
$X^\Box$: all the attributes in which the objects outside ***X*** do not possess
$X^\Diamond$: all the attributes which are carried by any member of ***X***

$$X_1 \subseteq X_2 \iff (X_2)^I \subseteq (X_1)^I$$

$$X_1 \subseteq X_2 \iff (X_1)^\Box \subseteq (X_2)^\Box$$

$$X_1 \subseteq X_2 \iff (X_1)^\Diamond \subseteq (X_2)^\Diamond$$

Galois Connection:
Objects and Attributes can be ordered simultaneously!

Concept := Node = 2-Tuple

The 2-tupples (X, X^I), $(X, X^\Box)$, $(X, X^\Diamond)$ can be ordered if $\underline{X}$ can be ordered!!

(The same relations hold for $Y \subseteq M$)

Y. Y. Yao, Concept lattices in rough set theory, Processing NAFIPS '04, IEEE Annual Meeting of the Fuzzy Information, Vol.2 (2004) 796-801

Rough Set Lattice

Not to be confused with Rough Set Theory!

(X, Y) is called an RSL concept if $X^{\Box} = Y$ and $Y^{\Diamond} = X$.

X is called the RSL extent and Y the RSL intent, e.g.,

$$\begin{cases} X = \{3,4,6\} \\ Y = \{b\} \end{cases} \text{gives} \begin{cases} X^{\Box} = \{b\} \\ Y^{\Diamond} = \{3,4,6\} \end{cases}$$

	a	b	c	d	e
1	X		X	X	X
2	X		X		
3		X			X
4		X			X
5	X				
6	X	X			X

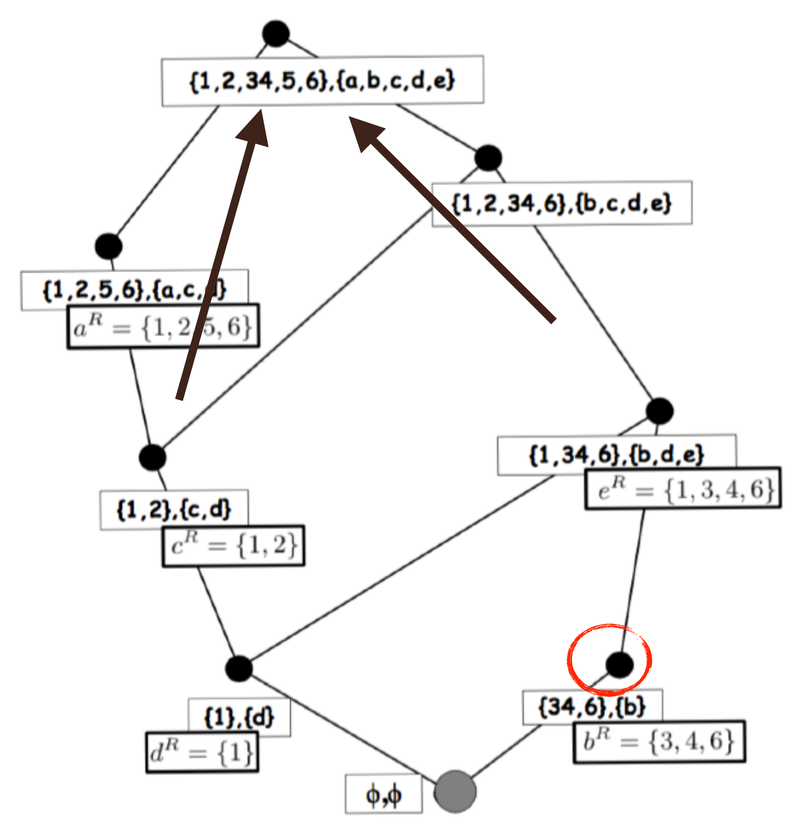
* Find **ALL** the particular Object-set Attribute-set pairs ***(X,Y)*** in which ***X*** (the extent) is the smallest object set having ***Y*** as peculiar property. These pairs can be ordered as a **Lattice** since

$$X_1 \subseteq X_2 \iff (X_1)^{\Box} \subseteq (X_2)^{\Box}$$

* $\forall Y \subseteq M \quad (Y^{\Diamond}, Y^{\Diamond\Box})$ is a RSL concept

* (X_1, Y_1) and (X_2, Y_2) are nodes $\Rightarrow (X_1 \cup X_2, Y_1 \cup Y_2)$ is a node,

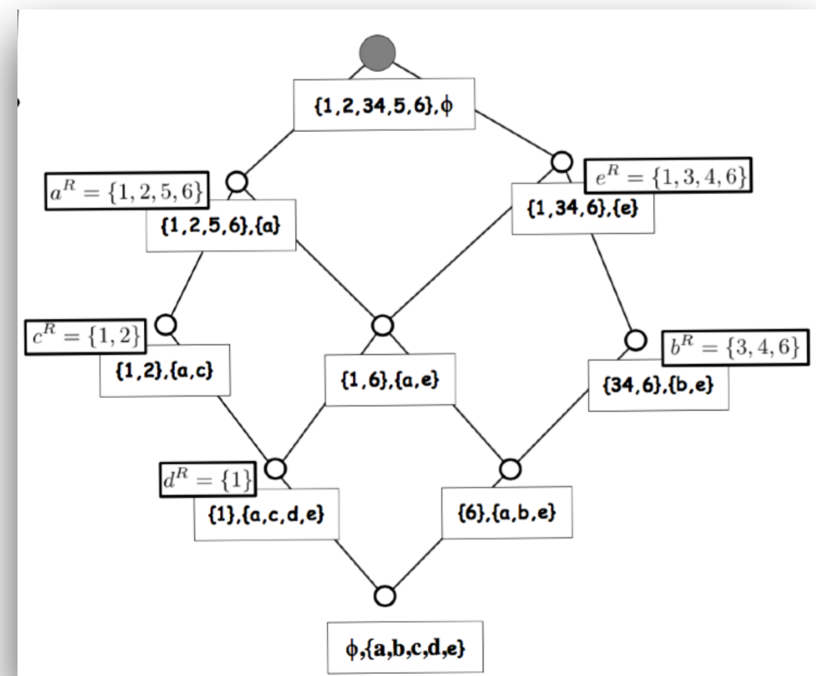
e.g., $\begin{cases} (\{1,3,4,6\},\{b,d,e\}) \\ (\{1,2,5,6\},\{a,c,d\}) \end{cases} \Rightarrow (\{1,2,3,4,5,6\},\{a,b,c,d,e\})$



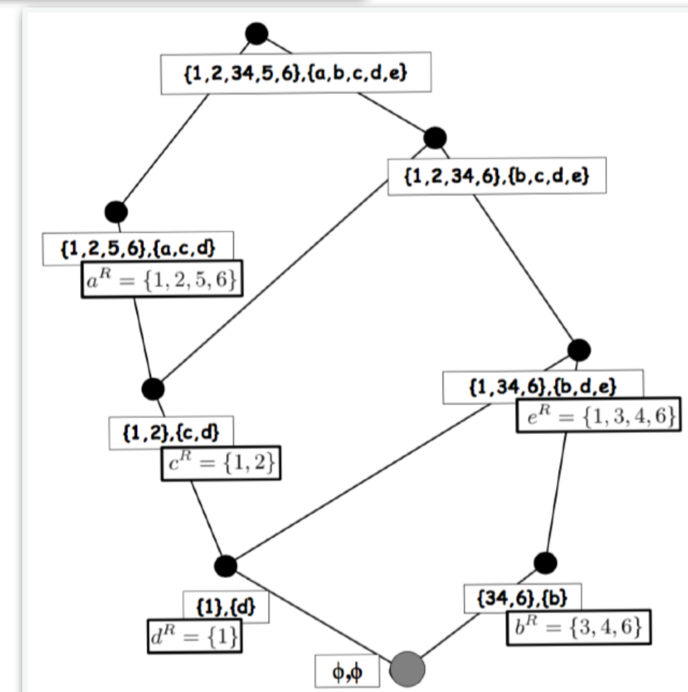
Issues with FCL and RSL

Given a formal context $F(G, M)$

$\forall m \in M$ the object-set m^R is a *common* extent for FCL and RSL



G \ M	a	b	c	d	e
1	X		X	X	X
2	X		X		
3		X			X
4		X			X
5	X				
6	X	X			X



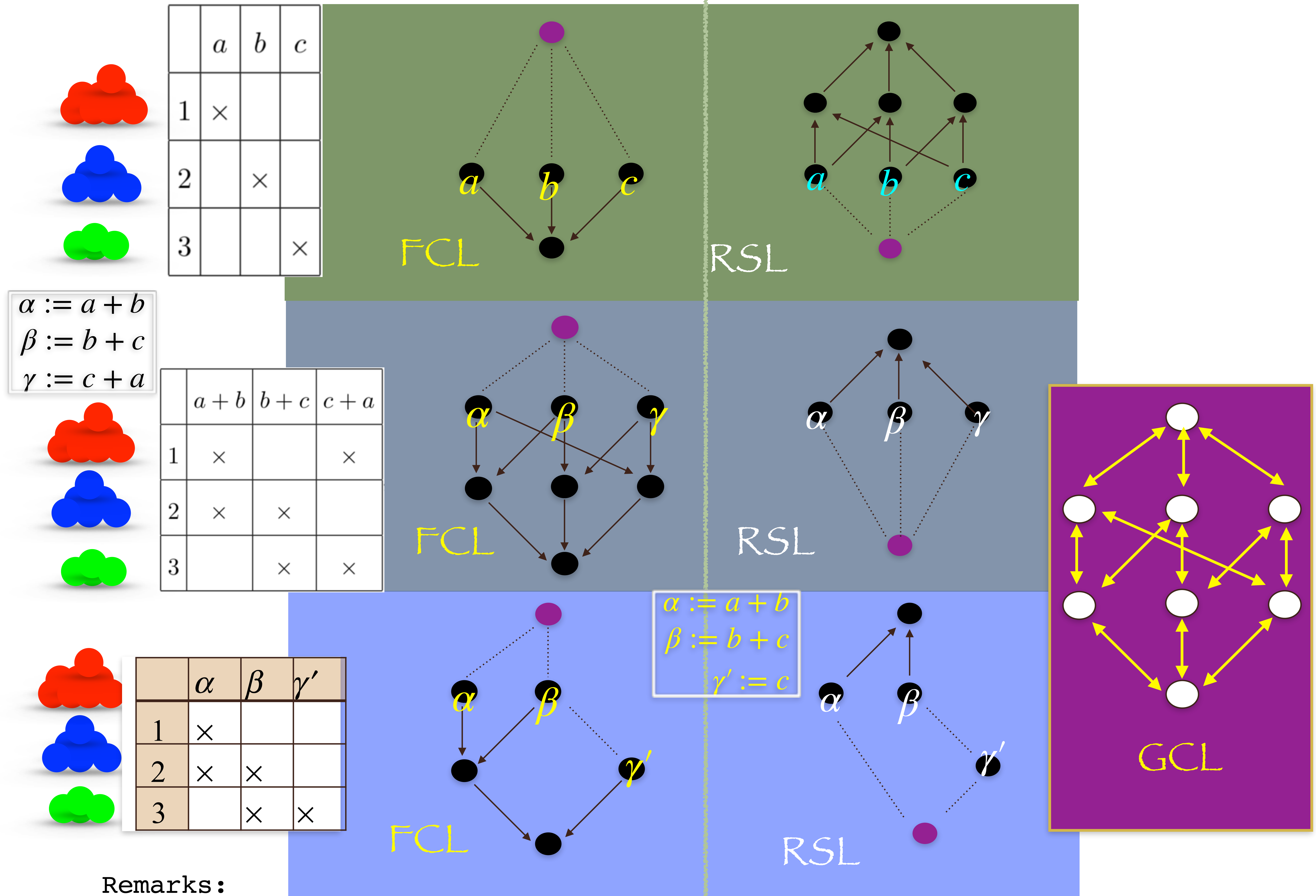
$(m^R, (m^R)^I)$ coincides with (Y^I, Y^{II}) and $(m^R, (m^R)^\Box)$ coincides with $(Y^\Diamond, Y^{\Diamond\Box})$.

Some node from FCL Do Not appear as node from RSL!

Not to mention how hard the Lattice Constructions could be, e.g.,

S.O. Kuznetsov, On Computing the Size of a Lattice and Related Decision Problems; Order 18, 4 (2001), 313-321.

S.O. Kuznetsov, S.A. Obiedkov, Comparing performance of algorithms for generating concept lattices; J. Exp. Theor. Artif. Intell. 14, 2-3 (2002), 189-216.



	a	b	c
1	×		
2		×	
3			×

	$a+b$	$b+c$	$c+a$
1	×		×
2	×	×	
3		×	×

	α	β	γ'
1	×		
2	×	×	
3		×	×

Remarks:

1. $\underline{a\bar{b}\bar{c} \equiv \alpha\bar{\beta}\gamma} = (a+b)\bar{b}\bar{c}(c+a)$, $\underline{\bar{a}b\bar{c} \equiv \alpha\beta\bar{\gamma}} = (a+b)(b+c)\overline{ca}$, $\underline{\bar{a}\bar{b}c \equiv \bar{\alpha}\beta\gamma} = \bar{a}\bar{b}(b+c)(c+a)$
2. $\underline{a\bar{b}\bar{c} \equiv \alpha\bar{\beta}\bar{\gamma}'} = (a+b)\bar{b}\bar{c}c$, $\underline{\bar{a}b\bar{c} \equiv \alpha\beta\bar{\gamma}'} = (a+b)(b+c)\bar{c}$, $\underline{\bar{a}\bar{b}c \equiv \bar{\alpha}\beta\gamma'} = \bar{a}\bar{b}(b+c)c$
3. ● has been added by hand.

Issues with FCL and RSL

Can We Consistently
Extend The List By Including More Columns
Without Altering The Information
Content ?

m^R is categorised as an object class in both the FCL and RSL just because m is listed in the table.

G \ M	a	b	c	d	e	b+c	ae
1	X		X	X	X	X	X
2	X		X			X	
3		X			X	X	
4		X			X	X	
5	X						
6	X	X			X	X	X

Formal Context
Vs
Truth Value Table

	a	b	c	d	e	b+c	ae
	T	F	T	T	T	T	T
	T	F	T	F	F	T	F
	F	T	F	F	T	T	F
	F	T	F	F	T	T	F
	T	F	F	F	F	F	F
	T	T	F	F	T	T	T

The new formal context tells nothing new even with new columns.
However, the new columns argue new nodes, while both FCL & RSL defy this.

$(b+c)^R = \{1,2,3,4,6\}$ gives a new FCL extent

$(a \cdot e)^R = \{1,6\}$ gives a new RSL extent

, where

$b+c$ stands for b OR c

$a \cdot e$ stands for a AND e

These Attributes Are Composite And Thus Should Not Be Adopted For The Categorisation !

GCL is based on Generalized Attributes

Given a set M of attributes, the set M^* of Generalised Attributes refer to the Composite Attributes one may construct out of M by means of the operations

$$\{\text{Conjunction}(\cdot),\text{Disjunction}(+),\text{Negation}(\neg)\}$$

All the M^* members must be referred to in the Categorisation!!

* The formal context that can be interpreted as a Truth Value table readily furnishes such reference.

The Formal Context $F(G,M)$ at hand is only a concise table structure, which can induce the complete information represented by the Extended Formal Context $F^*(G,M^*)$.

The Gcl Is To Be Based On F*(G,M*)

GCL node becomes 3-Tuple

Extended to F*(G, M*), M* are all the possible composite attributes

* $\forall\mu\ \mu^R$ is a `common extent` for RSL and FCL, referred to as the `General Extent` $X=\mu^R$

* μ^Rs exhaust all the possible General Extents; one obtains the `General Concept` as $\left(X,\rho(X),\eta(X)\right)$

$$\eta(X)=\prod X^{I^*} \qquad \rho(X)=\sum X^{\Box^*} \qquad (\rho(X))^R=(\eta(X))^R=X$$

- The `General Concepts` are well ordered as

$$X_1\subseteq X_2 \Leftrightarrow \begin{cases}\rho(X_1)\le\rho(X_2)\\ \eta(X_1)\le\eta(X_2)\end{cases} \Leftrightarrow \left(X_1,\rho(X_1),\eta(X_1)\right)\le\left(X_2,\rho(X_2),\eta(X_2)\right),$$ thereby forming the General Concept Lattice (GCL).

- The pair $\underline{\rho(X),\eta(X)}$ plays the role of `General Intent` $\left(\forall X\ \eta(X)\le\rho(X)\right)$, $\begin{cases}\rho(X) \text{ is the generalisation of RSL intent}\\ \eta(X) \text{ is the generalisation of FCL intent}\end{cases}$.
- The GCL is self-dual :
  $\left(X,\rho(X),\eta(X)\right)$ is a node (general concept) $\Rightarrow$ $\left(X,\rho(X),\eta(X)\right)^\dagger:=\left(X^c,\rho(X^c),\eta(X^c)\right)$ is also a node.

Construct GCL explicitly

$\eta(X) = \prod X^{I^*}$ $\rho(X) = \sum X^{\Box^*}$

$(\rho(X))^R = (\eta(X))^R = X$

The `General Concept` $(X,\rho(X),\eta(X)) \equiv \left(\underbrace{X}_{\text{general extent}}, \underbrace{[X]_F}_{\text{general intent}} \right)$,

$[X]_F \coloneqq \left\{ \mu \in M^* \middle| \mu^R = X \right\} \equiv \left\{ \mu \in M^* \middle| \eta(X) \le \mu \le \rho(X) \right\}$: $[X]_F$ is the closed interval $[\eta(X),\rho(X)]$.

- $\begin{cases} \text{The upper bound } \rho(X) = \sum_{\text{members in } [X]_F} [X]_F \in [X]_F \\ \text{The lower bound } \eta(X) = \prod_{\text{members in } [X]_F} [X]_F \in [X]_F \end{cases}$.

- $\begin{cases} \forall X_1 \in E_F \forall X_1 \in E_F \quad X_1 \ne X_2 \Leftrightarrow [X_1]_F \cap [X_2]_F = \varnothing \\ \forall X_i \in E_F \quad \bigcup_i X_i = M^* \end{cases}$,

$D_1 = \{1\}$
$D_2 = \{2\}$
$D_3 = \{3,4\}$
$D_4 = \{5\}$
$D_5 = \{6\}$

G \ M	a	b	c	d	e
1	X		X	X	X
2	X		X		
3		X			X
4		X			X
5	X				
6	X	X			X

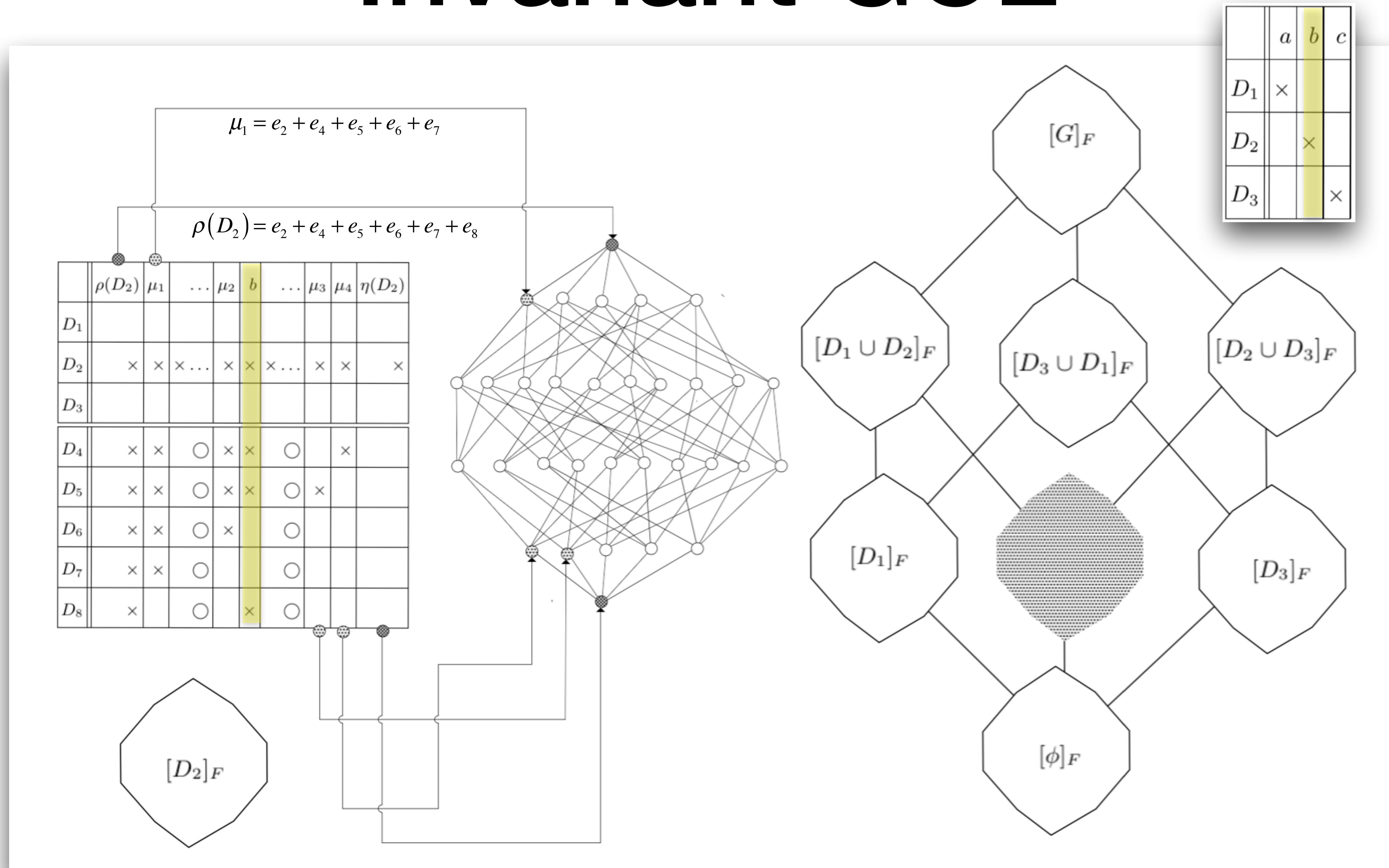
where $E_F = \{\text{all the general extents}\}$ can be constructed out

via all the possible unions of the smallest discernible object sets: $X = \bigcup_k D_k$ i.e. $\bigcup_{D_k \subseteq X} D_k$

$G_{/R} \coloneqq \{D_1, D_2, \ldots D_{n_F}\}$ $(|G_{/R}| = n_F)$, $E_F = \{\cup\wp | \wp \subseteq G_{/R}\}$ $(|E_F| = 2^{n_F}$, which is the number of nodes$)$

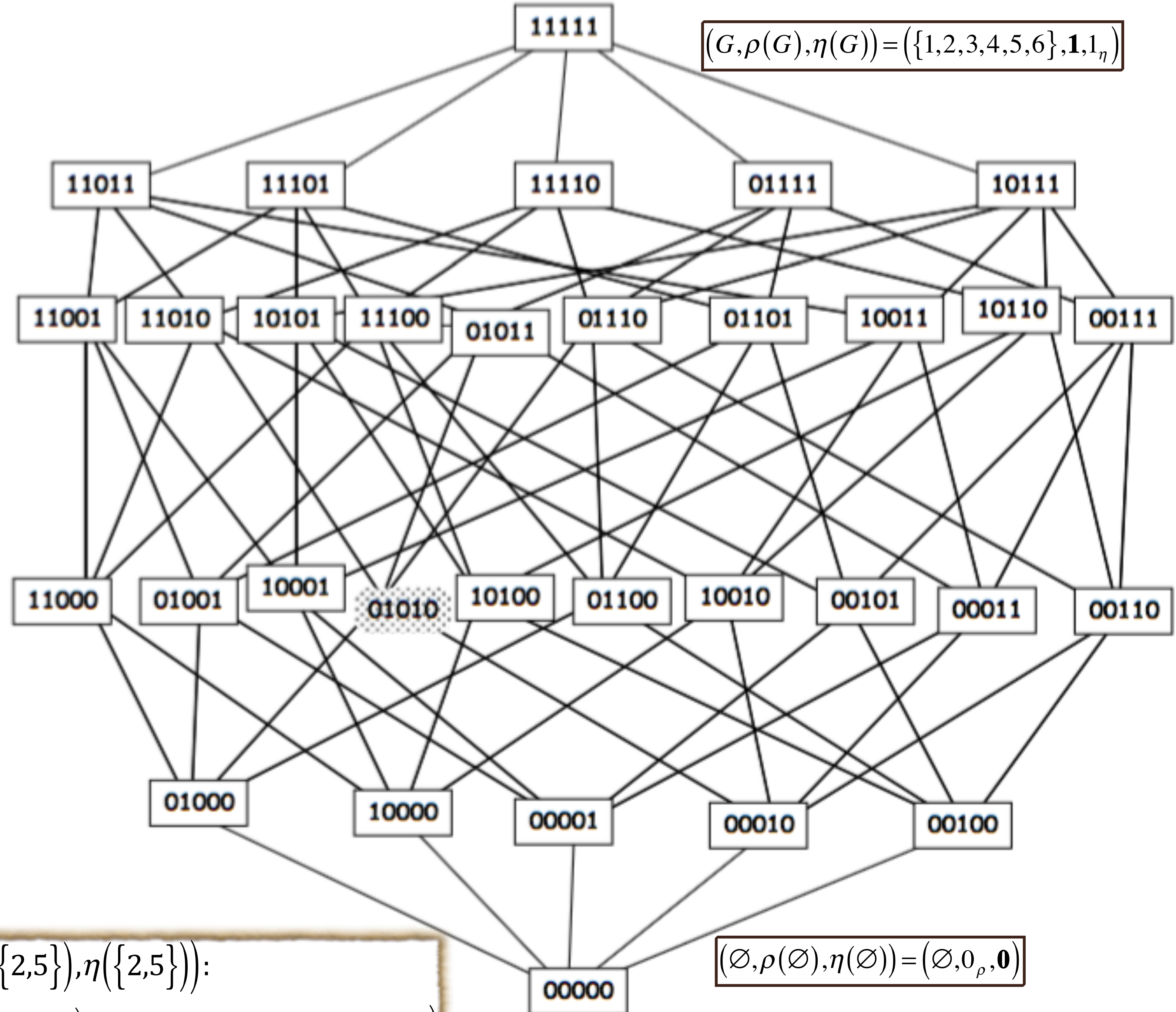
- $X \in E_F \Leftrightarrow X^c \coloneqq G \setminus X \in E_F$ with $\begin{cases} \rho(X^c) = \neg\eta(X) \\ \eta(X^c) = \neg\rho(X) \end{cases}$; no need to calculate both!

Invariant GCL



	a	b	c	d	e
1	×		×	×	×
2	×		×		
3		×			×
4		×			×
5	×				
6	×	×			×

$1_\eta =$
$a\neg bcde$ — $D_1 = \{1\}$
$+a\neg bc\neg d\neg e$ — $D_2 = \{2\}$
$+\neg ab\neg c\neg de$
$+a\neg b\neg c\neg d\neg e$ — $D_4 = \{5\}$
$+ab\neg c\neg de$ — $D_5 = \{6\}$



$$\left(G,\rho(G),\eta(G)\right)=\left(\{1,2,3,4,5,6\},\mathbf{1},1_\eta\right)$$

$$\left(\varnothing,\rho(\varnothing),\eta(\varnothing)\right)=\left(\varnothing,0_\rho,\mathbf{0}\right)$$

$$\left(D_2\cup D_4,\rho\left(D_2\cup D_4\right),\eta\left(D_2\cup D_4\right)\right)=\left(\{2,5\},\rho\left(\{2,5\}\right),\eta\left(\{2,5\}\right)\right):$$

$$\left(\{2,5\},\ \left(\neg a+b+\neg c+d+\neg e\right)\left(a+\neg b+c+d+\neg e\right)\left(\neg a+\neg b+c+d+\neg e\right),\ a\neg bc\neg d\neg e+a\neg b\neg c\neg d\neg e\right)$$

$$1_\eta = a\neg bcde + a\neg bc\neg d\neg e + \neg ab\neg c\neg de + a\neg b\neg c\neg d\neg e + ab\neg c\neg de$$

01010

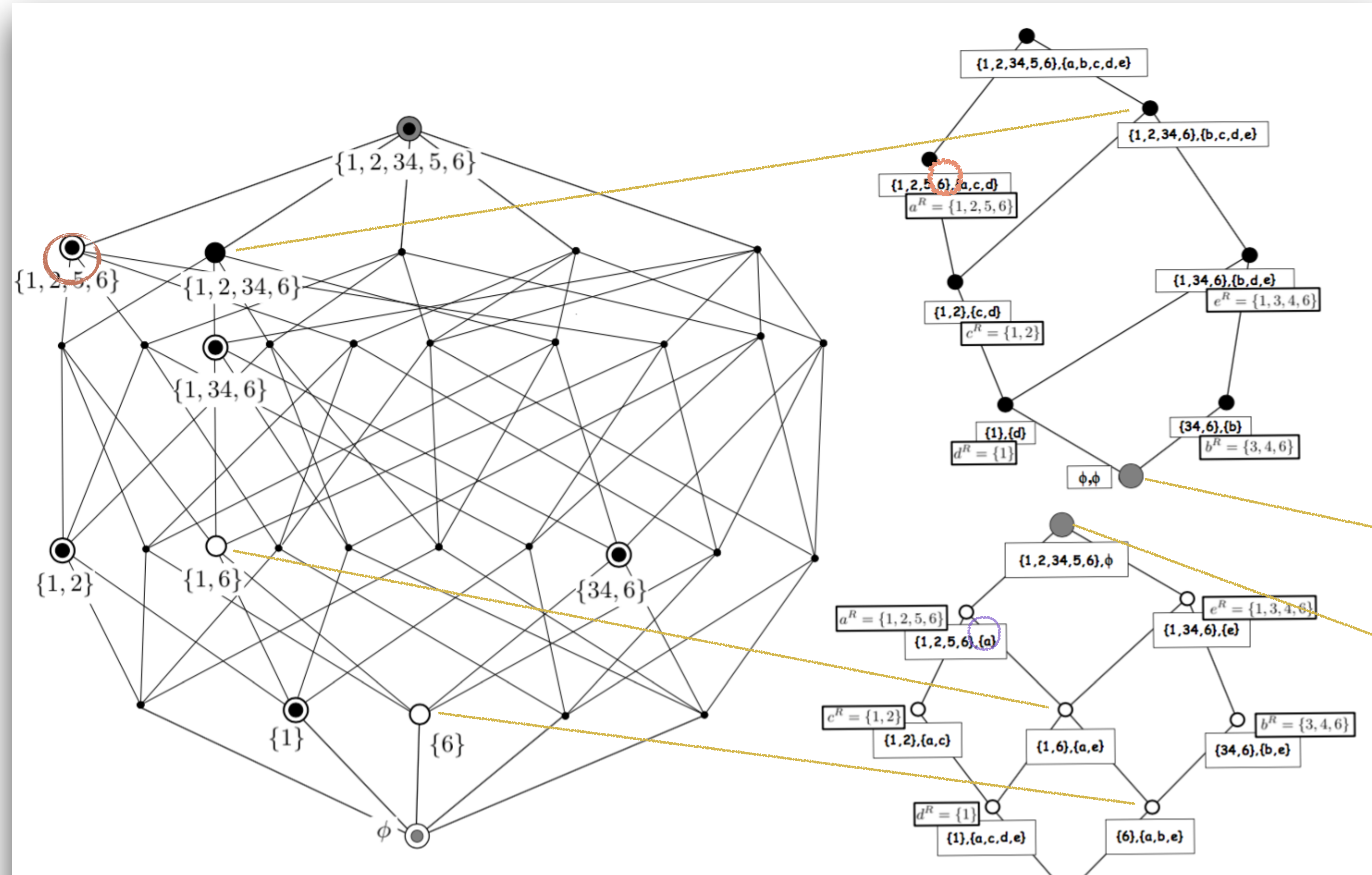
1 2 34 5 6

$$0_\rho = (\neg a+b+\neg c+\neg d+\neg e)(\neg a+b+\neg c+d+e)(a+\neg b+c+d+\neg e)(\neg a+b+c+d+e)(\neg a+\neg b+c+d+\neg e)$$

$$\eta(X)=\sum_{D_k\subset X}\eta(D_k)$$

$$\rho(X)=\neg\eta(X^c)=\prod_{D_k\subset X^c}\neg\eta(D_k)$$

Restore FCL and RSL from GCL



$a^R = \{1,2,5,6\}, b^R = \{3,4,6\}, c^R = \{1,2\}, d^R = \{1\}, e^R = \{1,3,4,6\}$

Artificial Completions of Lattices Which **Do Not Satisfy**

$X^{\Box} = Y$ and $Y^{\Diamond} = X$

Do Not Satisfy

$X^I = Y$ and $Y^I = X$

$$\begin{cases} \text{the FCL concept } (\{1,2,5,6\}, \{a\}) \\ \text{the RSL concept } (\{1,2,5,6\}, \{a,c,d\}) \end{cases}$$

$$\eta(\{1,2,5,6\}) = a\neg bcde + a\neg bc\neg d\neg e + a\neg b\neg c\neg d\neg e + ab\neg c\neg de$$

$$\equiv a(\neg b + e)(\neg d + e)(\neg b + \neg c)(\neg b + \neg d)(c + \neg d)(b + c + \neg e)(b + d + \neg e)(\neg c + d + \neg e)$$

$$\rho(\{1,2,5,6\}) = a + \neg b + c + d + \neg e$$

Logic Implication of GCL

$A \overset{FCL}{\to} B \iff A^I \subseteq B^I$

$\prod A \to \prod B$

$A \overset{RSL}{\to} B \iff A^\Diamond \subseteq B^\Diamond$

$\sum A \to \sum B$

$A^I \subseteq B^I \iff \mu_1^R \subseteq \mu_2^R$ with $\begin{cases} \mu_1 = \prod A \\ \mu_2 = \prod B \end{cases}$ $\because$ $\begin{cases} A^I = \bigcap_{m \in A} m^R = (\prod A)^R = \mu_1^R \\ B^I = \bigcap_{m \in B} m^R = (\prod B)^R = \mu_2^R \end{cases}$

$A^\Diamond \subseteq B^\Diamond \iff \mu_1^R \subseteq \mu_2^R$ with $\begin{cases} \mu_1 = \sum A \\ \mu_2 = \sum B \end{cases}$ $\because$ $\begin{cases} A^\Diamond = \bigcup_{m \in A} m^R = (\sum A)^R = \mu_1^R \\ B^\Diamond = \bigcup_{m \in B} m^R = (\sum B)^R = \mu_2^R \end{cases}$

Implications Of Gcl

$\forall \mu_1 \in M^* \forall \mu_2 \in M^* \quad \mu_1 \to \mu_2 \Leftrightarrow \mu_1{}^R \subseteq \mu_2{}^R$

General Concept $(X, \rho(X), \eta(X)) \equiv (X, [X]_F)$: General Extent X, General Intent $[X]_F = [\eta(X), \rho(X)]$

$\forall X \in E_F$

$\rho(X) \cdot 0_\rho = 0_\rho \quad \rho(X) + 0_\rho = \rho(X) \quad 0_\rho$: the Falsity for $\rho(X)$

$\eta(X) \cdot 1_\eta = \eta(X) \quad \eta(X) + 1_\eta = 1_\eta \quad 1_\eta$: the Truth for $\eta(X)$

Contextual Truth and Falsity

$\rho(G) = 1 \qquad \eta(G) = 1_\eta$

$\rho(\varnothing) = 0_\rho = \neg 1_\eta \quad \eta(\varnothing) = 0 = \neg 1$

With some $\mu \in M^*$, $(X, [\eta(X), \rho(X)]) = (\mu^R, [\mu \cdot 1_\eta, \mu + 0_\rho])$.

μ implies what

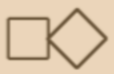
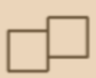
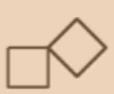
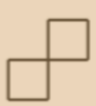
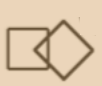
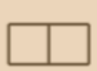
what implies μ

: which object class an object carrying μ should be categorised into ?

Answer: $\eta(X) \leq \mu \leq \rho(X)$ and $\mu^R = X$ which can be realised by $\begin{matrix} \mu \to \mu \cdot 1_\eta \\ \mu + 0_\rho \to \mu \end{matrix}$ $\left(\begin{matrix} \mu \leftrightarrow \mu \cdot 1_\eta \\ \mu + 0_\rho \leftrightarrow \mu \end{matrix} \right)$

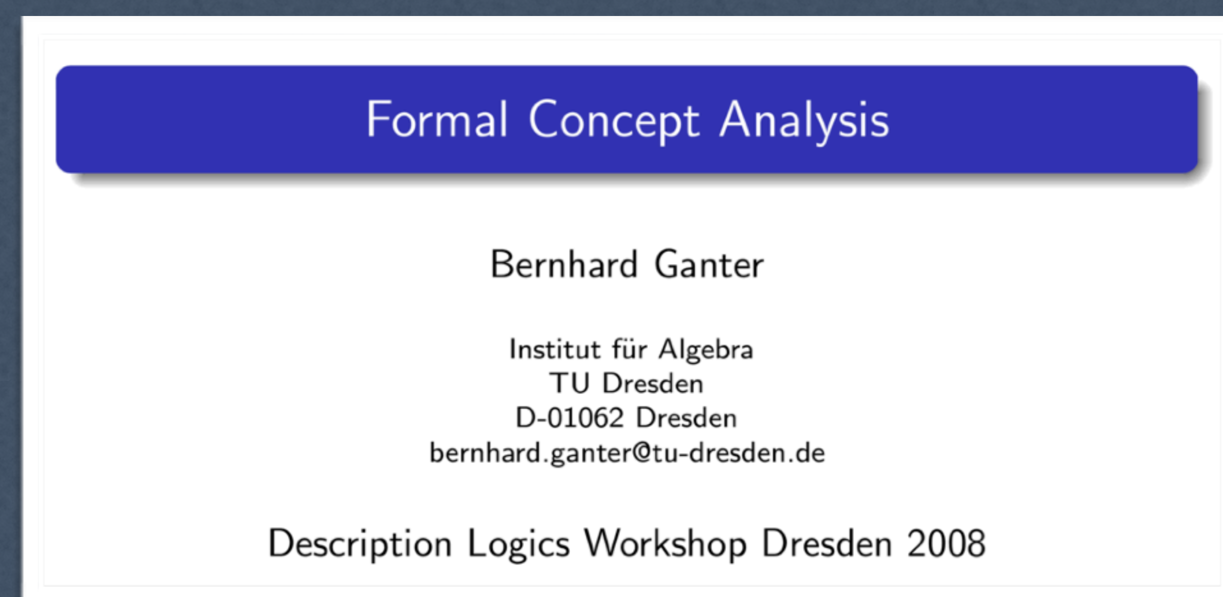
One rule is sufficient !!

$\forall \mu \in M^* \ \mu \to \mu \cdot 1_\eta$ is equivalent to $\forall \mu \in M^* \ \mu + 0_\rho \to \mu$

	o	d	p	s	e	v
						
		X				
		X	X			
	X		X			
	X		X	X	X	X
			X	X		
						X
			X			X
	X					
			X	X	X	X

o	overlap
d	disjoint
p	parallel
s	common segment
e	common edge
v	common vertex

Pairs of Squares Problem



$$1_\eta = \neg o \neg d \neg p \neg s \neg e \neg v + \neg o d \neg p \neg s \neg e \neg v + \neg o d p \neg s \neg e \neg v + o \neg d p \neg s \neg e \neg v + o \neg d p s e v$$

$$+ \neg o \neg d p s \neg e \neg v + \neg o \neg d \neg p \neg s \neg e v + \neg o \neg d p \neg s \neg e v + o \neg d \neg p \neg s \neg e \neg v + \neg o \neg d p s e v$$

$$1_\eta = \neg o \neg d \neg p \neg s \neg e \neg v + \neg o d \neg p \neg s \neg e \neg v + \neg o d p \neg s \neg e \neg v + o \neg d p \neg s \neg e \neg v + o \neg d p s e v$$
$$+ \neg o \neg d p s \neg e \neg v + \neg o \neg d \neg p \neg s \neg e v + \neg o \neg d p \neg s \neg e v + o \neg d \neg p \neg s \neg e \neg v + \neg o \neg d p s e v$$

- I. common edge → parallel, common vertex, common segment (e → p, v, s)
- II. common segment → parallel (s → p)
- III. parallel, common vertex, common segment → common edge
- IV. overlap, common vertex → parallel, common segment, common edge (o, v → p, s)
- V.
  - overlap, parallel, common segment → common edge, common vertex
  - overlap, parallel, common vertex → common segment, common edge
- VI.
  - disjoint, common vertex → ⊥
  - disjoint, parallel, common segment → ⊥
  - disjoint, overlap → ⊥

Working out these **stem bases** could take some efforts!

What one needs is only the formula

$$\forall \mu \in M^* \ \mu \to \mu \cdot 1_\eta$$

The FCL implications can be generated as below

I.

$$\underbrace{e}_{\text{common edge}} \to e \cdot 1_\eta = o \neg d p s e v + \neg o \neg d p s e \equiv \underbrace{\neg d p s v}_{\text{not disjoint, parallel, common segment, common vertex}} \ e$$

$$\{e\} \xrightarrow{FCL} \{p, s, v\}$$

IV.

$$ov \to ov \cdot 1_\eta = o \ \underbrace{\neg d p s e}_{\text{not disjoint, parallel, common seqment, common edge}} \ v$$

$$\{o, v\} \xrightarrow{FCL} \{p, s, e\}$$

II.

$$\underbrace{s}_{\text{common segment}} \to s \cdot 1_\eta = o \neg d p s e v + \neg o \neg d p s \neg e \neg v + \neg o \neg d p s e v \equiv \underbrace{(ev + \neg e \neg v)}_{\text{common edge} \leftrightarrow \text{common vertex}} \ \underbrace{\neg d p}_{\text{not disjoint, parallel}} \ s$$

III.

$$pvs \to pvs \cdot 1_\eta = o \neg d p s e v + \neg o \neg d p s e v = \neg d p s \underline{\underline{e}} v$$

V.

$$ops \to o \neg d p s \underline{\underline{e v}}$$
$$opv \to o \neg d p \underline{\underline{s}} e \underline{\underline{v}}$$

VI.

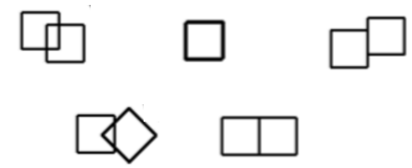
$$dv \to \mathbf{0}, \ dps \to \mathbf{0}, \ do \to \mathbf{0}$$

More than the FCL implications

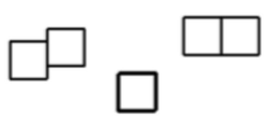
$$p+s+e \to p(\neg od\neg s\neg e\neg v + o\neg d\neg s\neg e\neg v + o\neg dsev + \neg o\neg ds\neg e\neg v + \neg o\neg d\neg s\neg ev + \neg o\neg dsev) \le p$$

an RSL implication : **OR** $\{p,s,e\} \xrightarrow{RSL} \{p\}$ $\left(\because (p+s+e)^R = p^R = \{\right.$  $\left.\}\right)$

$$o+s \to (o+s)\cdot 1_\eta = \neg d(psev + \neg ops\neg e\neg v + o\neg p\neg s\neg e\neg v) \le (p+o)$$

an RSL implication : $\{o,s\} \xrightarrow{RSL} \{o,p\}$ $\left(\because (o+s)^R = \{\right.$  $\left.\} \subset (o+p)^R\right)$

$$s \to s\cdot 1_\eta = s\neg dp(ev + \neg o\neg e\neg v) \le p$$

Both $\{s\} \xrightarrow{RSL} \{p\}$, $\{s\} \xrightarrow{FCL} \{p\}$ $\because s^R = \{$  $\} \subset p^R$, namely, $\{s\}^\Diamond \subset \{p\}^\Diamond$, $\{s\}^I \subset \{p\}^I$

$$\neg p \to \neg p\cdot 1_\eta = \neg o\neg d\neg p\neg s\neg e\neg v + \neg od\neg p\neg s\neg e\neg v + \neg o\neg d\neg p\neg s\neg ev + o\neg d\neg p\neg s\neg e\neg v$$

$$= \neg p\neg s\neg e(\neg o\neg d\neg v + \neg od\neg v + \neg o\neg dv + o\neg d\neg v)$$

Still something else

$$p\neg o + sv \to (p\neg o + sv)\cdot 1_\eta = \neg o\neg dps\neg e\neg v + \neg o\neg dp\neg s\neg ev + \neg o\neg dpsev + o\neg dpsev$$

$$= \neg dp(\neg os\neg e\neg v + \neg o\neg s\neg ev + sev)$$

$\mu_1 \xrightarrow{?} \mu_2$:

The theory of GCL suggests

$\mu_1 \to \mu_2 \Leftrightarrow \mu_1^R \subseteq \mu_2^R \Leftrightarrow \eta(\mu_1^R) \le \eta(\mu_2^R)$ OR $\rho(\mu_1^R) \le \rho(\mu_2^R)$

i.e. $\mu_1\cdot 1_\eta \le \mu_2\cdot 1_\eta$ OR $\mu_1 + 0_\rho \le \mu_2 + 0_\rho$ where $0_\rho \equiv \neg 1_\eta$

However, indeed, $\underbrace{\mu_1\cdot 1_\eta \le \mu_2\cdot 1_\eta \Leftrightarrow \mu_1 + 0_\rho \le \mu_2 + 0_\rho}_{\because \forall\mu\ \mu+0_\rho \equiv \mu\cdot 1_\eta + 0_\rho}$

Alterantively, use 1_η to check whether any implication is true

For example,

V.

$ops \to ev$

$\because ops\cdot 1_\eta = o\neg dpsev \le \neg dpsev = ev\cdot 1_\eta$

A Swimming-Race Puzzle

Five competitors –– A, B, C, D, and E –– enter a swimming race that awards gold, silver, and bronze medals to the first three to complete it. Each of the following compound statements about the race is *false*, although one of the two clauses in each *may* be true.

- A didn't win the gold, and B didn't win the silver.
- D didn't win the silver, and E didn't win the bronze.
- C won a medal, and D didn't.
- A won a medal, and C didn't.
- D and E both won medals.

Who won each of the medals?

http://www.rinkworks.com/brainfood/p/discrete5.shtml

The PDS in NS resolves a problem based on suggested parametrizations, which could be **non-unique** and **tedious**

the set of competitors be $\mathcal{Y} := \{A, B, C, D, E\}$

the set of medals be $\mathcal{M} = \{\mathbf{g}\text{old}, \mathbf{s}\text{ilver}, \mathbf{b}\text{ronze}\}$

y_m the competitor $y \in \mathcal{Y}$ won the medal $m \in \mathcal{M}$ (e.g. $A_\mathbf{s}$ means A won the silver medal)

$y = \sum_{m \in \mathcal{M}} y_m$ the competitor y won some medals

$\mathcal{Y}^\mathcal{M} = \{y_m \,|\, y \in \mathcal{Y}, m \in \mathcal{M}\}$

$m = \sum_{y \in \mathcal{Y}} y_m$ the medal m was won by some competitors.

choice made by Dave Wu 2013

The PDS in NS

The Swimming-Race Puzzle

The Given False Statements

1	A didn't win the gold, and B didn't win the silver.	$A_{\mathbf{g}} + B_{\mathbf{s}}$, i.e., $\neg A_{\mathbf{g}} \neg B_{\mathbf{s}} \leftrightarrow \mathbf{0}$
2	D didn't win the silver, and E didn't win the bronze.	$D_{\mathbf{s}} + E_{\mathbf{b}}$ $(\neg D_{\mathbf{s}} \neg E_{\mathbf{b}} \leftrightarrow \mathbf{0})$
3	C won a medal, and D didn't.	$\neg C + D$ $(C \neg D \leftrightarrow \mathbf{0})$
4	A won a medal, and C didn't.	$\neg A + C$ $(A \neg C \leftrightarrow \mathbf{0})$
5	D and E both won medals.	$\neg D + \neg E$ $(DE \leftrightarrow \mathbf{0})$

$\mathcal{Y} := \{A, B, C, D, E\}$

$\mathcal{M} = \{\mathbf{gold}, \mathbf{silver}, \mathbf{bronze}\}$

$\mathcal{Y}^{\mathcal{M}} = \{y_m \mid y \in \mathcal{Y}, m \in \mathcal{M}\}$

Some Implicit Conditions

6	each competitor obtained at most one of the medals	$\prod_{y \in Y} (\lvert y_{\mathbf{g}} \rvert y_{\mathbf{s}} \rvert y_{\mathbf{b}} \rvert)$
7	one medal was not given twice	$\prod_{m \in \mathcal{M}} (A_m \rvert B_m \rvert C_m \rvert D_m \rvert E_m)$
8	all the three medals were awarded out	$\prod_{m \in \mathcal{M}} (A_m + B_m + C_m + D_m + E_m)$

$$(3,4,5) \iff (\neg C + D)(\neg A + C)(\neg D + \neg E)$$

$$= CD\neg E + \neg C \neg A \neg D + \neg C \neg A \neg E$$

impossible to award all the medals out (contradicts $\underline{8}$)

$$(3,4,5,\underline{8,6}) \iff CD\neg E \prod_{m \in \mathcal{M}} (A_m + B_m + C_m + D_m + E_m) \cdot \prod_{y \in Y} (\lvert y_{\mathbf{g}} \rvert y_{\mathbf{s}} \rvert y_{\mathbf{b}} \rvert)$$

E obtained no medal $\therefore D_s$ by $\underline{2}$

$$\begin{cases} \prod_{y \in Y} (\ ) = \prod_{y \neq D} (\ ) \cdot (\lvert D_g \rvert D_s \rvert D_b \rvert) \\ D_g (\lvert D_g \rvert D_s \rvert D_b \rvert) = D_g \bar{D}_s \bar{D}_b \end{cases}$$

$$(3,4,5,8,6,\underline{2}) \iff CD_{\mathbf{s}} \neg E (A_{\mathbf{g}} + B_{\mathbf{g}} + C_{\mathbf{g}})(A_{\mathbf{b}} + B_{\mathbf{b}} + C_{\mathbf{b}}) \prod_{y \neq D} (\lvert y_{\mathbf{g}} \rvert y_{\mathbf{s}} \rvert y_{\mathbf{b}} \rvert) \cdot (\neg D_{\mathbf{g}} \neg D_{\mathbf{b}})$$

picks up only $A_g (B_b + C_b)$ by $\underline{1}$

$$(3,4,5,8,6,2,\underline{1,7}) \iff A_{\mathbf{g}} C D_{\mathbf{s}} \neg E (B_{\mathbf{b}} + C_{\mathbf{b}}) \prod_{y \neq D, A} (\lvert y_{\mathbf{g}} \rvert y_{\mathbf{s}} \rvert y_{\mathbf{b}} \rvert) \cdot (\neg D_{\mathbf{g}} \neg D_{\mathbf{b}})(\neg A_{\mathbf{s}} \neg A_{\mathbf{b}}) \cdot (A_{\mathbf{b}} \rvert B_{\mathbf{b}} \rvert C_{\mathbf{b}} \rvert D_{\mathbf{b}} \rvert E_{\mathbf{b}})(\neg B_{\mathbf{g}} \neg C_{\mathbf{g}})(\neg A_{\mathbf{s}} \neg B_{\mathbf{s}} \neg C_{\mathbf{s}})$$

$$\equiv A_{\mathbf{g}} D_{\mathbf{s}} C_{\mathbf{b}} (\neg A_{\mathbf{s}} \neg A_{\mathbf{b}})(\neg D_{\mathbf{g}} \neg D_{\mathbf{b}})(\neg C_{\mathbf{g}} \neg C_{\mathbf{s}}) \neg B \neg E$$

And so forth !!

A won the Gold Medal
D the Silver
C the Bronze

The PDS in NS as a problem solver

The allowable parametrisation is not unique: expert's formulation could be less intuitive; intuitive construction could be less concise.

Predicate(Subject) ≡ Attribute(Object):

the choice of parametrisation $\mathcal{Y} := \{A, B, C, D, E\}$ $\mathcal{M} = \{\text{gold}, \text{silver}, \text{bronze}\}$ $\mathcal{Y}^{\mathcal{M}} = \{y_m \mid y \in \mathcal{Y}, m \in \mathcal{M}\}$

suggests implicit objects, which are referred to as all the possible cases one may encouter :

$$\frac{A_{\mathbf{g}} + B_{\mathbf{s}} \quad D_{\mathbf{s}} + E_{\mathbf{b}} \quad \neg C + D \quad \neg A + C \quad \neg D + \neg E \quad \prod_{y\in Y}(|y_{\mathbf{g}}|y_{\mathbf{s}}|y_{\mathbf{b}}|) \quad \prod_{m\in\mathcal{M}}(A_m|B_m|C_m|D_m|E_m) \quad \prod_{m\in\mathcal{M}}(A_m + B_m + C_m + D_m + E_m)}{\text{(all cases)}}$$

$$= \frac{A_{\mathbf{g}}D_{\mathbf{s}}C_{\mathbf{b}}(\neg A_{\mathbf{s}}\neg A_{\mathbf{b}})(\neg D_{\mathbf{g}}\neg D_{\mathbf{b}})(\neg C_{\mathbf{g}}\neg C_{\mathbf{s}})\neg B\neg E}{\text{(all cases)}}$$

Choose the competitors as objects

(Surely, one may also choose the medals as objects)

consider the 5-tupple

$$\begin{bmatrix} \text{property of } A & \text{property of } B & \text{property of } C & \text{property of } D & \text{property of } E \end{bmatrix}$$

Explicit objects are also possible!!

- Entities that are individuals are suitable candidates for the objects.
- Employing the parametrisation that comprises more objects,

More PDSs in NS on different objects are to be dealt with in parallel (component-wise), Simultaneous consistency on every object (component) must be ensured.

intuitive but tedious!!

$$\ldots\underbrace{\left(\begin{bmatrix} 1 & 1 & 1 & s & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & b \end{bmatrix}\right)}_{\text{rule 2}}\underbrace{\left(\begin{bmatrix} 1 & 1 & \bar{g}\bar{s}b & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 & g+s+b \end{bmatrix}\right)}_{\text{rule 3}}\ldots\underbrace{(g\bar{s}\bar{b} + \bar{g}s\bar{b} + \bar{g}\bar{s}b)}_{\text{rule 6}}\ldots$$

$$= \begin{bmatrix} g\bar{s}\bar{b} & \bar{g}\bar{s}\bar{b} & \bar{g}\bar{s}b & \bar{g}s\bar{b} & \bar{g}\bar{s}\bar{b} \end{bmatrix}$$

A won the Gold Medal C the Bronze D the Silver

The PDS in NS

as a problem solver

Einstein's Puzzle
Variations of this riddle appear on the net from time to time.

Who keeps fish?

The Englishman lives in the red house.
The Swede keeps dogs.
The Dane drinks tea.
The green house is just to the left of the white one.
The owner of the green house drinks coffee.
The Pall Mall smoker keeps birds.
The owner of the yellow house smokes Dunhills.
The man in the center house drinks milk.
The Norwegian lives in the first house.
The Blend smoker has a neighbor who keeps cats.
The man who smokes Blue Masters drinks bier.
The man who keeps horses lives next to the Dunhill smoker.
The German smokes Prince.
The Norwegian lives next to the blue house.
The Blend smoker has a neighbor who drinks water.

https://web.stanford.edu/~laurik/fsmbook/examples/Einstein'sPuzzle.html

Convention

N Nationality	n_1 England	n_2 Sweden	n_3 Denmark	n_4 Germany	n_5 Norway
C Colour of house	c_1 Red	c_2 Green	c_3 White	c_4 Yellow	c_5 Blue
T Beverage	t_1 Tea	t_2 Coffee	t_3 Milk	t_4 Beer	t_5 Water
P Pet	p_1 Dog	p_2 Bird	p_3 Cat	p_4 Horse	p_5 (Fish)
S Brand of cigarettes	s_1 Pall Mall	s_2 Dunhill	s_3 Blue Masters	s_4 Prince	s_5 Blend

Choose the houses as objects: [property of h_1 | property of h_2 | property of h_3 | property of h_4 | property of h_5]

Solving The Einstein Riddle

Two Versions Concerning Which Is The First House

1	the Brit lives in the red house		$n_1c_1 + \neg n_1\neg c_1$ (i.e. $n_1 \leftrightarrow c_1$)
2	the Swede keeps dogs as pets		$n_2p_1 + \neg n_2\neg p_1$ ($n_2 \leftrightarrow p_1$)
3	the Dane drinks tea		$n_3t_1 + \neg n_3\neg t_1$ ($n_3 \leftrightarrow t_1$)
4	the green house is on the left of the white house		$[c_2, c_3, \mathbf{1}, \mathbf{1}, \mathbf{1}] + [\mathbf{1}, c_2, c_3.\mathbf{1}, \mathbf{1}] + \dots$
5	the green house's owner drinks coffee		$c_2t_2 + \neg c_2\neg t_2$ ($c_2 \leftrightarrow t_2$)
6	the Pall Mall smoker keeps birds		$s_1p_2 + \neg s_1\neg p_2$ ($s_1 \leftrightarrow p_2$)
7	the owner of the yellow house smokes Dunhill		$c_4s_2 + \neg c_4\neg s_2$ ($c_4 \leftrightarrow s_2$)
8	the man living in the center house drinks milk		$[\mathbf{1}, \mathbf{1}, t_3, \mathbf{1}, \mathbf{1}]$
9	the Norwegian lives in the first house	$v_\rightarrow$	$[n_5, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}]$
		$v_\leftarrow$	$[\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, n_5]$
10	the Blend smoker lives next to the one who		$[p_3, s_5, \mathbf{1}, \mathbf{1}, \mathbf{1}] + [\mathbf{1}, p_3, s_5, \mathbf{1}, \mathbf{1}] + [\mathbf{1}, \mathbf{1}, p_3, s_5, \mathbf{1}] + [\mathbf{1}, \mathbf{1}, \mathbf{1}, p_3, s_5]$ +
11	the man who keeps h lives next to the Dun		
12	the man who smokes		
13	the German smokes I		
14	the Norwegian lives i		
15	the Blend smoker has a neighbour who		

16*	the nationality is unique	$(n_1 \vert n_2 \vert n_3 \vert n_4 \vert n_5)$
16**		$\forall n \in N\ [n, \neg n, \neg n, \neg n, \neg n] + [\neg n, n, \neg n, \neg n, \neg n] + \dots$
17*	the colour is unique	$(c_1 \vert c_2 \vert c_3 \vert c_4 \vert c_5)$
17**		$\forall c \in C\ [c, \neg c, \neg c, \neg c, \neg c] + [\neg c, c, \neg c, \neg c, \neg c] + \dots$
18*	the beverage is unique	$(t_1 \vert t_2 \vert t_3 \vert t_4 \vert t_5)$
18**		$\forall t \in T\ [t, \neg t, \neg t, \neg t, \neg t] + [\neg t, t, \neg t, \neg t, \neg t] + \dots$
19*	the pet is unique	$(p_1 \vert p_2 \vert p_3 \vert p_4 \vert p_5)$
19**		$\forall p \in P\ [p, \neg p, \neg p, \neg p, \neg p] + [\neg p, p, \neg p, \neg p, \neg p] + \dots$
20*	the type of cigarettes is unique	$(s_1 \vert s_2 \vert s_3 \vert s_4 \vert s_5)$
		$\forall s \in S\ [s, \neg s, \neg s, \neg s, \neg s] + [\neg s, s, \neg s, \neg s, \neg s] + \dots$

implicit conditions!!

The uniqueness:

1. no one has two nationalities (colour-, beverage-, pet-, cigarette types).
2. no two have the same nationality, and so forth.

Multiplying together (component−wise) gives rise to the solutions for $v_\rightarrow$ and $v_\leftarrow$:

in both cases $c_2n_4\ p_5\ s_4t_2$ (but different houses). (fish)

Short-handed: $c_i \equiv c_i\prod_{j\neq i}\bar{c}_j,\ n_i \equiv n_i\prod_{j\neq i}\bar{n}_j,\ p_i \equiv p_i\prod_{j\neq i}\bar{p}_j,\ s_i \equiv s_i\prod_{j\neq i}\bar{s}_j,\ t_i \equiv t_i\prod_{j\neq i}\bar{t}_j.$

$v_\rightarrow$ $[c_4n_5p_3s_2t_5, c_5n_3p_4s_5t_1, c_1n_1p_2s_1t_3, c_2n_4p_5s_4t_2, c_3n_2p_1s_3t_4]$

$v_\leftarrow$ $[c_2n_4p_5s_4t_2, c_3n_2p_1s_3t_4, c_1n_1p_2s_1t_3, c_5n_3p_4s_5t_1, c_4n_5p_3s_2t_5]$